On the Monge-Ampère equation via prestrained elasticity

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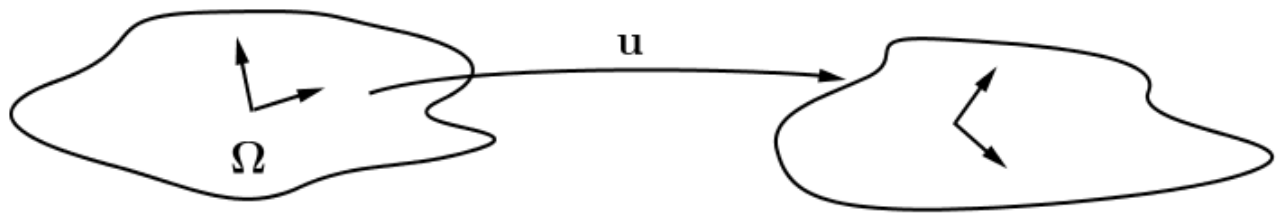
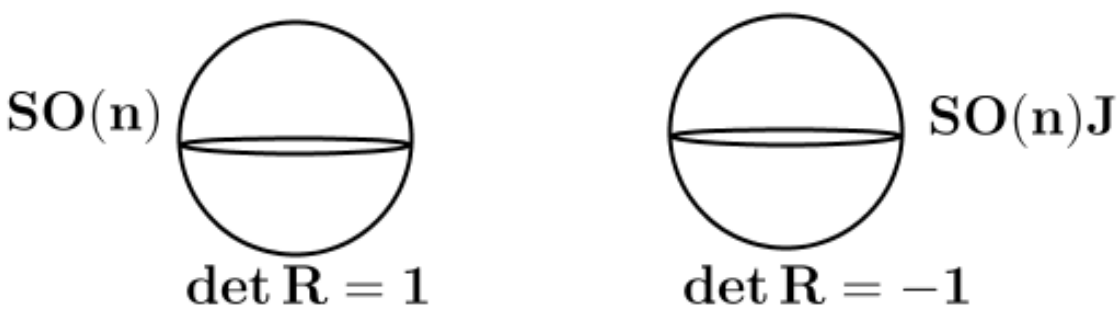
An old story: isometric immersions (equidimensional)

Assume that $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$ satisfies: $\nabla u(x)^T \nabla u(x) = Id_n$

- Equation of isometric immersion: $\langle \partial_i u, \partial_j u \rangle = \delta_{ij} = \langle e_i, e_j \rangle$
  (For $u \in \mathcal{C}^1$, this is equivalent to u preserving length of curves)



- Equivalent to: $\nabla u \in O(n) = \{R;\ R^T R = Id\} = SO(n) \cup SO(n)J$



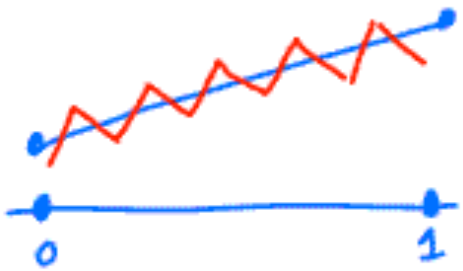
$J = \mathrm{diag}\{-1, 1, \ldots, 1\}$

- Liouville (1850), Reshetnyak (1967): $u \in W^{1,\infty}$ and $\nabla u \in SO(n)$ a.e. in $\Omega \Rightarrow \nabla u \equiv const \ \Rightarrow\ u(x) = Rx + b$ rigid motion

An old story: isometric immersions (equidimensional)

- Gromov (1973): Convex integration:
  $\exists u \in W^{1,\infty}$ such that $(\nabla u)^T \nabla u = Id$ a.e. in Ω, and ∇u takes values in $SO(n)$ and in $SO(n)J$, in every open $U \subset \Omega$.

Even more: $\exists u$ arbitrarily close to any u_0 with $0 < (\nabla u_0)^T \nabla u_0 < Id$

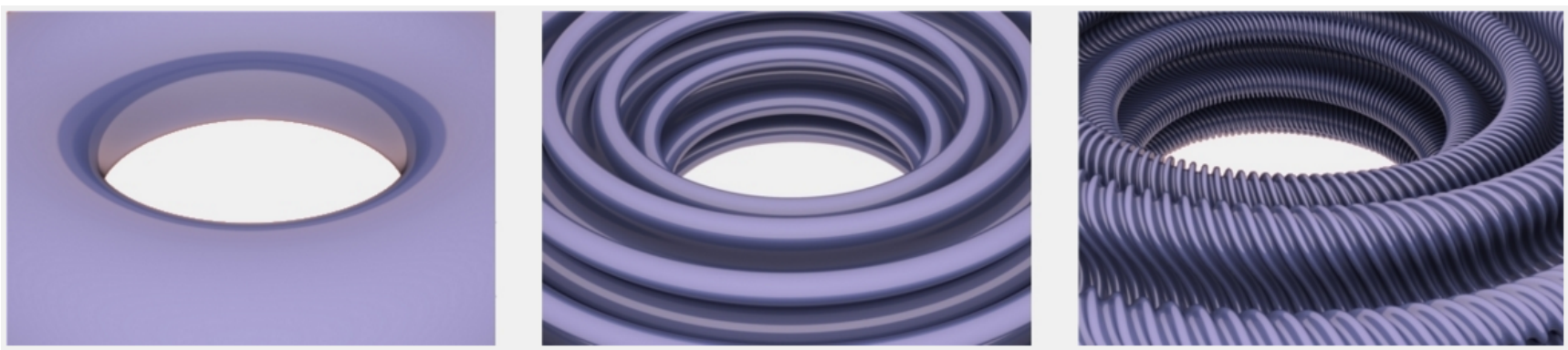


Example:

Given $u_0 : (0,1) \to \mathbb{R}$ with $(u_0')^2 < 1$

want: $u_k \xrightarrow{\text{uniformly}} u_0$ with $(u_k')^2 = 1$

more oscillations as $k \to \infty$



Hevea project: Inst. Camille Jordan, Lab J. Kuntzmann, Gipsa-Lab (France)

Isometric immersions of Riemann manifold (Ω, G)

Let $G \in \mathcal{C}^\infty(\Omega, \mathbb{R}^{n\times n}_{sym,+})$. Look for $u : \Omega \to \mathbb{R}^n$ so that $(\nabla u)^T \nabla u = G$ in Ω

Theorem (Gromov 1986)

Let $u_0 : \Omega \to \mathbb{R}^n$ be smooth short immersion, i.e.: $0 < (\nabla u_0)^T \nabla u_0 < G$ in Ω. Then: $\forall \varepsilon > 0 \quad \exists u \in W^{1,\infty} \quad \|u - u_0\|_{\mathcal{C}^0} < \varepsilon$ and $(\nabla u)^T \nabla u = G$.

Theorem (Myers-Steenrod 1939, Calabi-Hartman 1970)

Let $u \in W^{1,\infty}$ satisfy $(\nabla u)^T \nabla u = G$ and $\det \nabla u > 0$ a.e. in Ω. (For example, $u \in \mathcal{C}^1$ enough). Then $\Delta_G u = 0$ and so u is smooth. In fact, u is unique up to rigid motions, and: $\exists u \Leftrightarrow Riem(G) \equiv 0$ in Ω.

$$\boxed{E(u) = \int_\Omega W\big((\nabla u) G^{-1/2}(x)\big)\, dx}$$

$W(F) \sim \mathrm{dist}^2(F, SO(3))$

$G \in \mathcal{C}^\infty(\Omega, \mathbb{R}^{3\times 3}_{sym,+})$ incompatibility metric tensor

- $E(u) = 0 \Leftrightarrow \nabla u(x) \in SO(3) G^{1/2}(x) \ \ \forall a.e.\ x$

  $\Leftrightarrow (\nabla u)^T \nabla u = G$ and $\det \nabla u > 0$

Non-Euclidean elasticity

Lemma (L-Pakzad 2009)

$$\inf_{u \in W^{1,2}} E(u) > 0 \Leftrightarrow Riem(G) \not\equiv 0.$$

Thin non-Euclidean plates: $\Omega = \Omega^h = \omega \times (-h/2, h/2), \quad \omega \subset \mathbb{R}^2$

- <u>As $h \to 0$:</u> Scaling of: $\inf E^h \sim h^\beta$? $\operatorname{argmin} E^h \to \operatorname{argmin} I_\beta$?
- Hierarchy of theories I_β, where β depends on $Riem(G^h)$
  Bhattacharya, Li, L., Mahadevan, Pakzad, Raoult, Schaffner
- When $G = Id$: dimension reduction in nonlinear elasticity
  seminal analysis by LeDret-Raoult 1995, Friesecke-James-Muller 2006

Manufacturing residually-strained thin films:

- *Shaping of elastic sheets by prescription of Non-Euclidean metrics* (Klein, Efrati, Sharon) Science, 2007
- *Half-tone gel lithography* (Kim, Hanna, Byun, Santangelo, Hayward) Science, 2012
- *Defect-activated liquid crystal elastomers* (Ware, McConney, Wie, Tondiglia, White) Science, 2015

The Monge-Ampère constrained energy

$$\text{Energy} \quad E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big((\nabla u^h)(G^h)^{-1/2}(x)\big)\,\mathrm{d}x$$

Theorem (L-Ochoa-Pakzad 2014)

Let: $G^h(x', x_3) = Id_3 + 2hS(x')$. *Then:*

- $\inf E^h \leq Ch^3 \;\Leftrightarrow\; \exists v \in W^{2,2}(\omega),\;\; \det \nabla^2 v = -\mathit{curl\ curl}\, S_{2\times 2}$
- $\frac{1}{h^3}E^h \xrightarrow{\Gamma} I$, *where I is the 2-d energy:*

$$I(v) = \int_\omega |\nabla^2 v|^2 \text{ for } v \in W^{2,2}(\omega),\;\; \det \nabla^2 v = -\mathit{curl\ curl}\, S_{2\times 2}$$

[More general result for $G^h(x', x_3) = Id_3 + 2h^\gamma S(x')$ and $\gamma \in (0,2)$.
When $\gamma \geq 2$ then higher order models.]

<u>Structure of minimizers to E^h:</u> $u^h(x', 0) = x' + h^{1/2} v e_3$

- $\kappa(\nabla(id + h^{1/2} v e_3)^T \nabla(id + h^{1/2} v e_3)) = \kappa(Id_2 + h\nabla v \otimes \nabla v)$

$$= -\tfrac{1}{2} h \,\mathrm{curl\ curl}\,(\nabla v \otimes \nabla v) + O(h^2) = h \det \nabla^2 v + O(h^2)$$

- Gauss curvature: $\kappa(Id_2 + 2hS_{2\times 2}) = -h\,\mathrm{curl\ curl}\, S_{2\times 2} + O(h^2)$

Weak formulation of the Monge-Ampère equation

$\det \nabla^2 v = f$ • existence of $W^{2,2}$ solutions is not guaranteed

$$\boxed{\text{Det } \nabla^2 v = -\tfrac{1}{2} \text{ curl curl}\big(\nabla v \otimes \nabla v\big)} \qquad v \in W^{1,2}(\omega)$$

Need to solve: $\text{curl curl } (\nabla v \otimes \nabla v) = \text{curl curl } S_{2\times 2}$

where $S_{2\times 2} = \lambda \mathrm{Id}_2$ with $\Delta\lambda = -2f$ in ω.

Equivalently: $\boxed{\nabla v \otimes \nabla v + \text{sym}\nabla w = S_{2\times 2}}$ 3 eqns in 3 unknowns on a 2d domain

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Similar problem 1: $(\nabla u)^T \nabla u = G_{2\times 2}$, where $u : \omega \to \mathbb{R}^3$ isometric immersion of 2d metric in $\mathbb{R}^3$.

- Nirenberg (1953): $\forall G_{2\times 2}$, $\kappa > 0$ $\exists$ smooth isometr. embed. in $\mathbb{R}^3$
- Poznyak-Shikin (1995): Same true for $\kappa < 0$ on bounded $\omega \subset \mathbb{R}^2$
- Nash-Kuiper (1956): $\forall n$-dim G $\exists \mathcal{C}^{1,\alpha}$ isometr. embed. in $\mathbb{R}^{n+1}$

Case $G_{2\times 2}$: Borisov (2004), Conti-Delellis-Szekelyhidi (2010) $\alpha < \frac{1}{7}$

Delellis-Inauen-Szekelyhidi (2015) $\alpha < \frac{1}{5}$.

- $\mathcal{C}^{1,\frac{2}{3}+}$ solutions are rigid – convex case: Borisov (2004).

Weak formulation of the Monge-Ampère equation

$\det \nabla^2 v = f$ • existence of $W^{2,2}$ solutions is not guaranteed

$$\boxed{\text{Det } \nabla^2 v = -\tfrac{1}{2} \text{ curl curl}\big(\nabla v \otimes \nabla v\big)} \qquad v \in W^{1,2}(\omega)$$

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Equivalently: $\boxed{\nabla v \otimes \nabla v + \text{sym}\nabla w = S_{2\times 2}}$ 3 eqns in 3 unknowns on a 2d domain

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Similar problem 2: $\partial_t u + \text{div}(u \otimes u) + \nabla p = 0, \quad \text{div}\, u = 0$

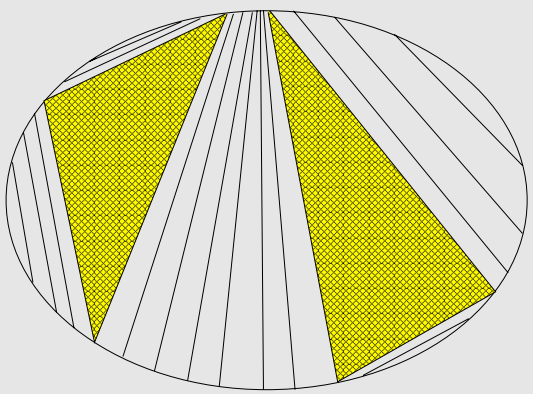
3d incompressible Euler equations, $(u,p) : \mathbb{T}^4 \times [0,T] \to \mathbb{R}^4$.

- Onsager's conjecture: rigidity/flexibility treshold $= \frac{1}{3}$.
- Constantin-E-Titi, Eyink (1994): Every $L^\infty(0,T;\mathcal{C}^\alpha(\mathbb{T}^3))$ solution, $\alpha > \frac{1}{3}$, is energy conserving.
- Delellis, Szekelyhidi, Buckmaster, Isett: existence of non-energy-conserving solutions, for every $\alpha < \frac{1}{3}$: compactly supported in time; arbitrary temporal kinetic energy profile.

Rigidity for the Monge-Ampère equation

Theorem (L-Pakzad 2015)

Let $v \in \mathcal{C}^{1,\frac{2}{3}+}$. *If* $\mathrm{Det}\nabla^2 v = 0$, *then* v *is developable,* *i.e.* $\forall x \in \omega$:



- *either* v *is affine in* $B_\varepsilon(x)$
- *or* ∇v *is constant on a segment through* x, *joining* $\partial\omega$ *on both ends.*

If $\mathrm{Det}\nabla^2 v \geq c > 0$ *is Dini continuous, then* v *is locally convex and an Alexandrov solution in* ω.

Theorem (Pakzad 2004, Sverak 1991, L-Mahadevan-Pakzad 2013)

Let $v \in W^{2,2}$. *If* $\det\nabla^2 v = 0$ *then* v *is developable and* $v \in \mathcal{C}^1$.
If $\det\nabla^2 v \geq c > 0$ *then* $v \in \mathcal{C}^1$ *and* v *(or* $-v$*) is locally convex.*

Flexibility for the Monge-Ampère equation

Theorem (L-Pakzad 2015)

Let $(v_0, w_0) : \omega \to \mathbb{R} \times \mathbb{R}^2$ *be a smooth short infinitesimal, i.e.:*

$$\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0 < S_{2\times 2}.$$

Then $\exists (v_n, w_n) \in \mathcal{C}^{1,\frac{1}{7}-}$ $(v_n, w_n) \xrightarrow{\text{uniformly}} (v_0, w_0)$ *and*

$$\nabla v_n \otimes \nabla v_n + \mathrm{sym}\nabla w_n = S_{2\times 2}.$$

- Extension [L-Pakzad-Inauen 2017]: flexibility at $\mathcal{C}^{\frac{1}{5}-}$.

Corollary ("Ultimate flexibility")

Let $f \in L^{\frac{7}{6}}(\omega)$ *and* $\alpha < \frac{1}{7}$. *The set of* $\mathcal{C}^{1,\alpha}(\bar{\omega})$ *solutions to the Monge - Ampère equation:* $\mathrm{Det}\, \nabla^2 v = f$ *is dense in the space* $\mathcal{C}^0(\bar{\omega})$.

- For $f \in L^p(\omega)$ and $p \in (1, \frac{7}{6})$, the density holds for any $\alpha < 1 - \frac{1}{p}$.
- $\mathrm{Det}\, \nabla^2$ is weakly discontinuous everywhere in $W^{1,2}(\omega)$.

Consequences for energy scaling: flexibility at $\mathcal{C}^{1,\frac{1}{7}-} \Rightarrow \inf E^h \leq Ch^{\frac{9}{4}}$.

(If flexibility at $\mathcal{C}^{1,\frac{1}{3}-}$, optimal for Nash-Kuiper, $\Rightarrow \inf E^h \leq Ch^{\frac{10}{4}}$).

Energy gap: $Ch^{3-} \leq \inf E^h \leq Ch^{\frac{10}{4}+}$: residual energy? fine crumpling?

Rigidity for the Monge-Ampère equation

Tool: Degree formula via the commutator estimate.

Commutator estimate argument in the Euler rigidity:

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0, \quad \operatorname{div} u = 0.$$

- Mollify on scale ε: $\partial_t(u_\varepsilon) + \operatorname{div}(u \otimes u)_\varepsilon + \nabla p_\varepsilon = 0, \ \operatorname{div} u_\varepsilon = 0$
- Integrate by parts with u_ε: $\frac{d}{dt}\left(\int \frac{1}{2}|u_\varepsilon|^2\right) - \int \left\langle (u \otimes u)_\varepsilon : \nabla u_\varepsilon \right\rangle = 0$
- Add new trilinear term for free:

$$\frac{d}{dt}\left(\int \frac{1}{2}|u_\varepsilon|^2\right) = \int \left\langle (u \otimes u)_\varepsilon - (u_\varepsilon \otimes u_\varepsilon) : \nabla u_\varepsilon \right\rangle.$$

Use commutator estimate: $\|(fg)_\varepsilon - f_\varepsilon g_\varepsilon\|_{C^k} \leq C\varepsilon^{2\alpha - k}\|f\|_{C^{0,\alpha}}\|g\|_{C^{0,\alpha}}$

to bound: $\left|\int \left\langle (u \otimes u)_\varepsilon - (u \otimes u)_\varepsilon : \nabla u_\varepsilon \right\rangle\right| \leq C\varepsilon^{2\alpha}\varepsilon^{\alpha-1} = C\varepsilon^{3\alpha-1}$

$\to 0$ when $\alpha > \frac{1}{3}$.

For the Monge-Ampère equation, use similar argument in the geometrical context.

Rigidity for the Monge-Ampère equation

Lemma (L-Pakzad 2015)

Let $v \in C^{1,\frac{2}{3}+}$, $f \in L^{1+}$ *satisfy:* $\mathrm{Det}\nabla^2 v = f$. *Then:*

$$\int_U (\phi \circ \nabla v) f = \int_{\mathbb{R}^2} \phi(y) deg(\nabla v, U, y)\, \mathrm{d}y \qquad \forall\, \phi \in L^\infty(U \Subset \omega)$$
$$supp\phi \subset \mathbb{R}^2 \setminus (\nabla v)(\partial U).$$

<u>Proof.</u> $\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w = A, \quad f = -\mathrm{curl}\,\mathrm{curl}\, A.$

- Mollify on scale ε: $\frac{1}{2}(\nabla v \otimes \nabla v)_\varepsilon + \mathrm{sym}\nabla w_\varepsilon = A_\varepsilon$
- Apply degree formula to smooth v_ε:
  $\int_U (\phi \circ \nabla v_\varepsilon) \det \nabla^2 v_\varepsilon = \int_{\mathbb{R}^2} \phi(y) \mathrm{deg}(\nabla v_\varepsilon, U, y)\, \mathrm{d}y \to RHS$
- Error in LHS: $\int_U (\phi \circ \nabla v_\varepsilon) \det \nabla^2 v_\varepsilon - (\phi \circ \nabla v) f$
  $= \int (\phi \circ \nabla v_\varepsilon - \phi \circ \nabla v) f - \int (\phi \circ \nabla v_\varepsilon) \mathrm{curl}\,\mathrm{curl}\left(\frac{1}{2}\nabla v_\varepsilon \otimes \nabla v_\varepsilon - A\right)$
- first term $\to 0$, last term: $\int (\phi \circ \nabla v_\varepsilon) \mathrm{curl}\,\mathrm{curl}\left(\frac{1}{2}\nabla v_\varepsilon \otimes \nabla v_\varepsilon - A_\varepsilon\right)$
  $= \frac{1}{2}\int (\phi \circ \nabla v_\varepsilon) \mathrm{curl}\,\mathrm{curl}\left(\nabla v_\varepsilon \otimes \nabla v_\varepsilon - (\nabla v \otimes \nabla v)_\varepsilon\right)$
  $= -\frac{1}{2}\int \left\langle \nabla^\perp (\phi \circ \nabla v_\varepsilon), \mathrm{curl}\left(\nabla v_\varepsilon \otimes \nabla v_\varepsilon - (\nabla v \otimes \nabla v)_\varepsilon\right)\right\rangle$

  bounded by: $C\varepsilon^{\alpha-1}\varepsilon^{2\alpha-1} = C\varepsilon^{3\alpha-2} \qquad \to 0$ when $\alpha > \frac{2}{3}$. ■

Rigidity for the Monge-Ampère equation

Lemma (L-Pakzad 2015)

Let $v \in C^{1,\frac{2}{3}+}$, $f \in L^{1+}$ *satisfy:* $\mathrm{Det}\nabla^2 v = f$. *Then:*

$$\int_U (\phi \circ \nabla v) f = \int_{\mathbb{R}^2} \phi(y) deg(\nabla v, U, y)\, \mathrm{d}y \qquad \forall\, \phi \in L^\infty(U \Subset \omega)$$
$$supp\phi \subset \mathbb{R}^2 \setminus (\nabla v)(\partial U).$$

<u>Proof.</u> (...)

- $-\frac{1}{2}\int \left\langle \nabla^\perp(\phi \circ \nabla v_\varepsilon), \mathrm{curl}\big(\nabla v_\varepsilon \otimes \nabla v_\varepsilon - (\nabla v \otimes \nabla v)_\varepsilon\big)\right\rangle$

  bounded by: $C\varepsilon^{\alpha-1}\varepsilon^{2\alpha-1} = C\varepsilon^{3\alpha-2}$ $\qquad \to 0$ when $\alpha > \frac{2}{3}$.

Lemma (Friedlander-L-Pavlovic-Shvydkoy 2016)

The degree formula holds as well for: $v \in C^1 \cap B^{5/3}_{3,c_0}$, $f \in L^1$.

because the flux:

$$\int \left\langle \nabla^\perp(\phi \circ \nabla S_Q v), \mathrm{curl}\big(\nabla S_Q v \otimes \nabla S_Q v - S_Q(\nabla v \otimes \nabla v)\big)\right\rangle$$
$$\to 0 \text{ as } Q \to \infty.$$

Proof of rigidity

Developable case: Assume $f = 0$, so: $\deg(\nabla v, U, y) = 0 \ \ \forall y \notin \nabla v(\partial U)$.

- Need: $\mathrm{Int}\big(\nabla v(\omega)\big) = \emptyset$. Then Pogorelov results $\Rightarrow$ developab.
- $\exists$ Cesar-type construction (Maly-Martio) of $u : \mathbb{R}^2 \to \mathbb{R}^2$: $\deg(u, \cdot, \cdot) = 0$ but $u(\omega)$ has full measure!
- Main step: use gradient structure of $u = \nabla v$ to show $\forall U \Subset \omega$:

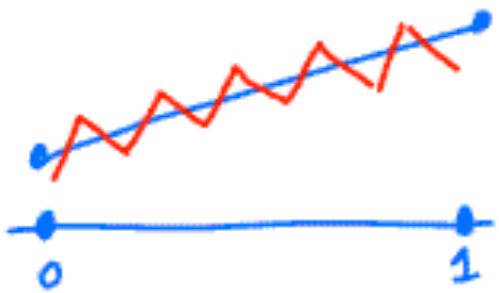
  $\nabla v(U) \subset \nabla v(\partial U)$, which has measure 0 by $\nabla v \in \mathcal{C}^{0,\frac{2}{3}+}$.

Convex case: Assume $f > 0$, so: $\deg(\nabla v, U, y) > 0 \ \ \forall y \notin \nabla v(\partial U)$.

- $\nabla v \in BV$ i.e.: $\displaystyle\sum_{i=1}^{N} |\nabla v(E_i)| \leq \int f < \infty \quad \forall \{E_i\}_{i=1}^{N} \in \omega$ disjoint.
- Graph(v) is a surface of bounded extrinsic curvature and its every regular point is elliptic. Then Pogorelov results $\Rightarrow$ v is convex or concave in a nbhd of any x regular
- The above sets ω_r^+ and ω_r^- are both open, and:

  $\forall x \notin \omega_r^+ \cup \omega_r^- \quad \exists$ line L from x to $\partial\omega$ with $\nabla v \equiv$ const on L.
- Conclusion: $\omega = \omega_r^+$ or $\omega = \omega_r^-$.

Convex integration for Monge-Ampère



Given $v_0 : (0,1) \to \mathbb{R}$ with $(v_0')^2 < 1$

want: $v_n \xrightarrow{\text{uniformly}} v_0$ with $(v_n')^2 = 1$

more oscillations as $n \to \infty$

- *Decomposition into primitive metrics.*

  Write: $\mathcal{A} = S_{2\times 2} - \big(\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0\big) > 0$

  Hence: $\mathcal{A}(x) = \sum_{k=1}^{3} \phi_k^2(x)\eta_k \otimes \eta_k$ where $\eta_1, \eta_2, \eta_3 \in \mathbb{S}^1$

- *Improvement by steps.*

  (v_0, w_0) given. Fix $\lambda > 0$. Find $(\bar{v}, \bar{w})$ so that:

$$(*) \quad \nabla\bar{v} \otimes \nabla\bar{v} + \mathrm{sym}\nabla\bar{w} = \big(\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0\big) + \phi_1^2(x)\eta_1 \otimes \eta_1 + O(\tfrac{1}{\lambda})$$

Let $\eta_1 = e_1$ and define: $\bar{v} = v_0 + \text{correction}, \quad \bar{w} = w_0 + \text{correction}$

Convex integration for Monge-Ampère

- (v_0, w_0) given. Fix $\lambda > 0$. Find $(\bar{v}, \bar{w})$ so that:

$$D = (\nabla\bar{v}\otimes\nabla\bar{v} + \mathrm{sym}\nabla\bar{w}) - (\nabla v_0\otimes\nabla v_0 + \mathrm{sym}\nabla w_0) = \boxed{\phi_1^2 e_1\otimes e_1 + O(\frac{1}{\lambda})}$$

Define: $\bar{v} = v_0 + \phi_1(x)\dfrac{V(\lambda x_1)}{\lambda}$

$$\bar{w} = w_0 - 2\phi_1(x)\frac{V(\lambda x_1)}{\lambda}\nabla v_0 + \phi_1^2(x)\frac{W(\lambda x_1)}{\lambda}e_1$$

Then: $\nabla\bar{v} = \nabla v_0 + \phi_1 V' e_1 + O(\frac{1}{\lambda})$

$$\nabla\bar{w} = \nabla w_0 - 2\phi_1 V' e_1\otimes\nabla v_0 + \phi_1^2 W' e_1\otimes e_1 + O(\tfrac{1}{\lambda})$$

$$D = \phi_1^2\underbrace{\left((V')^2 + W'\right)}_{\text{want = 1}} e_1\otimes e_1 + O(\tfrac{1}{\lambda}).$$

Take: $V(t) = \frac{1}{\sqrt{2\pi}}\sin(2\pi t)$

$W(t) = -\frac{1}{4\pi}\sin(4\pi t)$

- In 3 steps, improving by: $\phi_1^2(x)\eta_1\otimes\eta_1$, $\phi_2^2(x)\eta_2\otimes\eta_2$, $\phi_3^2(x)\eta_3\otimes\eta_3$, we get: $\nabla\bar{v}_\lambda\otimes\nabla\bar{v}_\lambda + \mathrm{sym}\nabla\bar{w}_\lambda = S_{2\times 2} + O(\frac{1}{\lambda})$

Improvement by stages

- *Improvement by stages:*

  $(\bar{v}_\lambda, \bar{w}_\lambda) \xrightarrow{\mathcal{C}^{1,\alpha}} (v_n, w_n)$ with: $\nabla v_n \otimes \nabla v_n + \mathrm{sym}\nabla w_n = S_{2\times 2}$

  and: $\|(v_n, w_n) - (v_0, w_0)\|_{\mathcal{C}^0} \leq \frac{1}{n}$

- Stage: Given $(\hat{v}, \hat{w})$ and $\lambda > 1$, we obtain $(\bar{v}, \bar{w})$ such that:

  calling $\hat{E} := \|\hat{\mathcal{A}}\|_0 = \|S_{2\times 2} - (\nabla \hat{v} \otimes \nabla \hat{v} + \mathrm{sym}\nabla \hat{w})\|_0$

  $\|\bar{v} - \hat{v}\|_0 + \|\bar{w} - \hat{w}\|_0 \leq C\frac{\hat{E}^{1/2}}{\lambda}$

  $\|\bar{v} - \hat{v}\|_1 + \|\bar{w} - \hat{w}\|_1 \leq C\hat{E}^{1/2} \Rightarrow \|\cdot\|_1 \sim \lambda^{-m/2}$

  $\|\bar{v} - \hat{v}\|_2 + \|\bar{w} - \hat{w}\|_2 \leq C(1 + \|\hat{v}\|_2 + \|\hat{w}\|_2)\lambda^3 \Rightarrow \|\cdot\|_2 \sim \lambda^{3m}$

  $\bar{E} := \|S_{2\times 2} - (\nabla \bar{v} \otimes \nabla \bar{v} + \mathrm{sym}\nabla \bar{w})\|_0 \leq C\frac{\hat{E}}{\lambda} \Rightarrow \bar{E}_m \sim \lambda^{-m}$

- Iterate and interpolate by: $\|\cdot\|_{0,\alpha} \leq \|\cdot\|_0^{1-\alpha} \| \cdot\|_1^{\alpha}$

  $$\Rightarrow \qquad \|\bar{v}_{m+1} - \bar{v}_m\|_{1,\alpha} \leq C\lambda^{-\frac{m}{2}(1-\alpha)}\,\lambda^{3m\alpha} = C\lambda^{(\frac{7}{2}\alpha - \frac{1}{2})m}$$

  Thus to get: $(\bar{v}_m, \bar{w}_m) \to (v, w)$ in $\mathcal{C}^{1,\alpha}$, we need: $\alpha < \frac{1}{7}$. ■

Improvement by stages

- (...) Iterate and interpolate by: $\|\cdot\|_{0,\alpha} \leq \|\cdot\|_0^{1-\alpha}\| \cdot\|_1^{\alpha}$

  $$\Rightarrow \qquad \|\bar{v}_{m+1} - \bar{v}_m\|_{1,\alpha} \leq C\lambda^{-\frac{m}{2}(1-\alpha)}\,\lambda^{3m\alpha} = C\lambda^{(\frac{7}{2}\alpha-\frac{1}{2})m}$$

  Thus to get: $(\bar{v}_m, \bar{w}_m) \to (v, w)$ in $\mathcal{C}^{1,\alpha}$, we need: $\alpha < \frac{1}{7}$.

- Better regularity expected if one can reduce the number of steps:

  2 steps in stage $\Rightarrow$ $\alpha < \frac{1}{5}$; 1 step $\Rightarrow$ $\alpha < \frac{1}{3}$.

- 2 steps: Replace $\mathcal{A}$ by a diagonal $\bar{\mathcal{A}}$, at each stage.

  $\mathcal{A} = S_{2\times 2} - (\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0) = \sum_{k=1}^{3} \phi_k^2 \eta_k \otimes \eta_k > 0$

  $\bar{\mathcal{A}} = S_{2\times 2} - (\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla(w_0 + \bar{w})) = \mathcal{A} - \mathrm{sym}\nabla\bar{w} = \phi^2 Id_2$,

  through: $\mathrm{curl}\,\mathrm{curl}\,\mathcal{A} = \mathrm{curl}\,\mathrm{curl}(g\,Id) = \Delta g$, $\|g\|_{\mathcal{C}^{0,\beta}} \leq C\|\mathcal{A}\|_{\mathcal{C}^{0,\beta}}$.

  Take: $\mathrm{sym}\nabla\bar{w} = \mathcal{A} - \bar{\mathcal{A}}$, $\phi^2 = g > 0$. Then:

$$\|\bar{w}\|_{\mathcal{C}^{1,\beta}} \leq C\left(\|\mathcal{A}\|_{\mathcal{C}^{0,\beta}} + \|\bar{\mathcal{A}}\|_{\mathcal{C}^{0,\beta}}\right) \leq C\left(\|\mathcal{A}\|_{\mathcal{C}^{0,\beta}} + \|g\|_{\mathcal{C}^{0,\beta}}\right) \leq C\|\mathcal{A}\|_{\mathcal{C}^{0,\beta}}.$$

Thank you for your attention